Linear Algebra I

13/04/2018, Friday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Linear equations

12 + 3 = 15 pts

Consider the linear equation

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 1 & -2 & 4 & 6 \\ -1 & 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ b-2a \end{bmatrix}$$

where a and b are real numbers.

a. Find all values of a and b for which the equation is consistent. For these values find the general solution of the equation.

b. Find all values of a and b for which the equation has a unique solution.

2 Partitioned matrices and matrix inverse

7 + 8 = 15 pts

Let

$$M = \begin{bmatrix} I & A \\ B & C \end{bmatrix}$$

where all four blocks are $n \times n$ matrices.

a. Show that M is nonsingular if and only if $C - BA$ is nonsingular.

b. Suppose that $C - BA$ is nonsingular. Find M^{-1}.

3 Row and column spaces

5 + 5 + 5 = 15 pts

Let $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times r}$, and $A = BC$. Show that

a. the column space of A is a subspace of the column space of B.

b. the row space of A is a subspace of the row space of C.

c. $\operatorname{rank}(A) \leqslant \min\{\operatorname{rank}(B), \operatorname{rank}(C)\}$.

4 Vector spaces

$(2+2)+(3+4+4) = 15$ pts

a. Let $M \in \mathbb{R}^{2\times 2}$ and consider the set $V_M = \{x \in \mathbb{R}^2 \mid x^T M x = 0\}$.

(i) Find a matrix M such that V_M is *not* a subspace of $\mathbb{R}^2$. Justify your answer.

(ii) Find a matrix M such that V_M is a subspace of $\mathbb{R}^2$. Justify your answer.

b. Let $y \in \mathbb{R}^2$ and consider the set $V_y = \{A \in \mathbb{R}^{2\times 2} \mid y^T A y = 0\}$.

(i) Show that V_y is a subspace $\mathbb{R}^{2\times 2}$.

(ii) Let $y = \begin{bmatrix}1 & 1\end{bmatrix}^T$. Show that

$$\left\{ \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix} \right\} \qquad (\star)$$

is a basis for V_y.

(iii) Let $L : V_y \to V_y$ be given by $L(A) = A^T$. Show that L is a linear transformation. Find its matrix representation relative to the basis given in $(\star)$.

5 Eigenvalues and diagonalization

$3+6+6 = 15$ pts

Consider the matrix

$$\begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}.$$

a. Find its determinant.

b. Find its eigenvalues.

c. Is it diagonalizable? Justify your answer. If it is so, find a diagonalizer.

6 Bases for vector spaces

$5+10 = 15$ pts

Let $E = (v_1, v_2, v_3)$ be an ordered basis for the vector space V.

a. Find the dimension of V. Justify your answer.

b. Let a, b, c be scalars. Find all values of a, b, c such that the vectors

$$v_1 + av_2,\ v_2 + bv_3,\ v_3 + cv_1$$

form a basis for V.

10 pts free